A Lower Bound for the Distributed Lovász Local Lemma

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Outline

1. The Lovász local lemma
2. Our model of distributed computing
3. A lower bound for the distributed LLL

The probabilistic method

- Primarily used in combinatorics to give existence proofs.
- Randomly choose objects from a certain class, and show that the probability that the object is of the desired kind is larger than zero.
- It follows that at least one such object has to exist.

The probabilistic method

- Primarily used in combinatorics to give existence proofs.
- Randomly choose objects from a certain class, and show that the probability that the object is of the desired kind is larger than zero.
- It follows that at least one such object has to exist.

- Let $\mathcal{E} = \{E_1, \ldots, E_n\}$ be a set of *bad* events that make the object undesirable.
- If the events are mutually independent and $\Pr(E_i) < 1$ for each i, we have trivially $\Pr(\bigcap_{i=1}^{n} \overline{E_i}) > 0$.
- What if there is some dependence between the events?

The Lovász local lemma (LLL)

Theorem (Erdős and Lovász, 1975)

Let $\mathcal{E} = \{E_1, \ldots, E_n\}$ be a finite set of events such that each E_i depends on at most d other events. If $\Pr(E_i) \leq p$ and $4pd \leq 1$, then there is a positive probability that none of the events occur.

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Theorem (Lovász, 1977)

Let $\mathcal{E} = \{E_1, \ldots, E_n\}$ be a finite set of events such that each E_i depends on at most d other events. If $\Pr(E_i) \leq p$ and $ep(d+1) \leq 1$, then there is a positive probability that none of the events occur.

$e = 2.718\ldots$ is the base of the natural logarithm.

LLL: an example

Proposition

Any instance ϕ of k-SAT where no variable appears in more than $\frac{2^{k-2}}{k}$ clauses is satisfiable.

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Proof.

1. Pick a truth assignment uniformly at random.
2. Let E_i denote the event "clause i is not satisfied".
3. $\Pr(E_i) = 2^{-k} =: p$.
4. E_i depends on at most $d := k\frac{2^{k-2}}{k} = 2^{k-2}$ other events.
5. We have $4pd = 4 \cdot 2^{-k} \cdot 2^{k-2} = 1$.
6. Now LLL implies that $\Pr(\bigcap \overline{E_i}) > 0$.

□

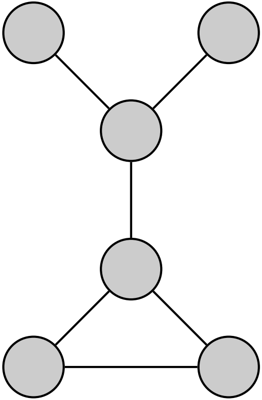
The algorithmic LLL

- LLL itself does not give a method for finding the object whose existence it proves.

- Beck showed in 1991 that there exist a deterministic polynomial-time algorithm for a weaker variant of LLL.
- This inspired a long line of reseach about algorithms for various versions of LLL.
- The breakthrough result of Moser and Tardos (2010) shows that there is a simple randomised resampling algorithm for a very general form of LLL.

- But we are interested in the *distributed* algorithmic LLL.

Distributed computing: the LOCAL model

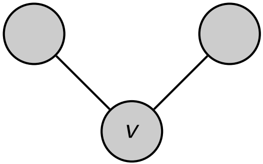


A simple connected undirected graph $G = (V, E)$, where each node $v \in V$

- is given its own input,
- runs the same algorithm,
- communicates with its neighbours,
- produces its own output.

Communication in synchronous rounds

Initially, each node v knows the total number of nodes n, the maximum degree of the graph Δ, and a task-specific local input $f(v)$.



In every round, each node $v \in V$

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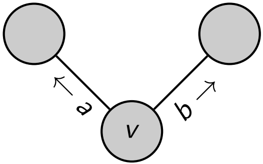

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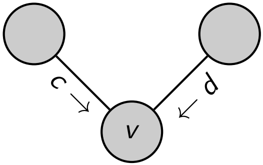

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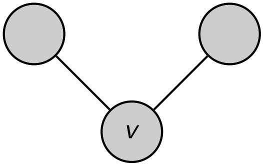

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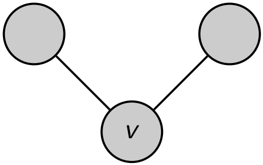

In every round, each node $v \in V$

1. sends messages to its neighbours,
2. receives messages from its neighbours,
3. updates its state.

After the final round, each node announces its output.

The running time of an algorithm is the *number of communications rounds* until all nodes have stopped, as a function of n.

Input and output

- The same graph $G = (V, E)$ serves both as the communication network and as the *problem instance*.
- A *graph problem* is defined by a function Π that maps each graph G and each labelling $f: V \to X$ to a set $\Pi(G, f)$ of *solutions* $S: V \to Y$.

- The *output* of algorithm A in (G, f) is the function $g: V \to Y$ such that $g(v)$ is the local output of v for each node v.
- Algorithm A *solves* problem Π if for each graph G and labelling f the output g of A in (G, f) is in $\Pi(G, f)$.

Randomised algorithms

- We assume that each node can toss a countably infinite number of random coins.
- Equivalently, each node v is given a real number $x(v)$ taken uniformly at random from $[0, 1]$.
- With probability 1, the values $x(v)$ are globally unique and can thus be used as identifiers.

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- Monte Carlo algorithms:
  - Running time is deterministic.
  - Output is a valid solution with high probability (with probability at least $1 - 1/n^c$ for an arbitrarily large constant c).

The distributed Lovász local lemma

- Let $\mathcal{X} = \{X_1, \ldots, X_m\}$ be a set of mutually independent random variables and let $\mathcal{E} = \{E_1, \ldots, E_n\}$ be a set of events.
- Denote by $\mathrm{vbl}(E_i) \subseteq \mathcal{X}$ the subset of variables that E_i depends on.
- Define a *dependency graph* $G_{\mathcal{E}} = (\mathcal{E}, \mathcal{D})$, where $\mathcal{D} = \{\{E_i, E_j\} : \mathrm{vbl}(E_i) \cap \mathrm{vbl}(E_j) \neq \emptyset\}$.

Problem

Let the communication network be isomorphic to $G_{\mathcal{E}} = (\mathcal{E}, \mathcal{D})$; each node v corresponds to an event $E_v \in \mathcal{E}$ and knows the set $\mathrm{vbl}(E_v)$. The task is to have each node v output an assignment a_v of the variables $\mathrm{vbl}(E_v)$ such that

1. *for any $\{E_u, E_v\} \in \mathcal{D}$ and $X \in \mathrm{vbl}(E_u) \cap \mathrm{vbl}(E_v)$ it holds that $a_u(X) = a_v(X)$,*
2. *the event E_v does not occur under assignment a_v.*

Existing algorithms and lower bounds

- The algorithm of Moser and Tardos (2010) can be adapted to the distributed setting; the running time is $O(\log^2 n)$ rounds.
- Chung et al. (2014) gave a distributed algorithm running in $O(\log n)$ rounds in bounded-degree graphs.

- LLL can be used to properly colour a cycle graph using a constant number of colours. This is known to require $\Omega(\log^* n)$ rounds.

Our lower bound

Theorem

Let $f : \mathbb{N} \to \mathbb{R}$ be such that $f(4) \leq 16$. Let A be a Monte Carlo distributed algorithm for LLL that finds an assignment avoiding all the bad events under the LLL criteria $pf(d) \leq 1$ with high probability. Then the running time of A is $\Omega(\log \log n)$ rounds.

Note that we can plug in, for example, either of the LLL criteria $ep(d+1) \leq 1$ or $4pd \leq 1$.

Outline of the proof

- Two new graph problems: *sinkless orientation* and *sinkless colouring*.
- LLL can be used to solve the sinkless orientation in 3-regular graphs.
- A *mutual speedup lemma*:
  - If we can find a sinkless colouring in t rounds, we can find a sinkless orientation in t rounds.
  - If we can find a sinkless orientation in t rounds, we can find a sinkless colouring in $t - 1$ rounds.
- By iterating the lemma, we obtain an algorithm that finds a sinkless orientation in 0 rounds, which leads to a contradiction.

Orientations

- An *orientation* σ of a graph $G = (V, E)$ assigns a direction

$$\sigma(\{u, v\}) \in \{u \to v, u \leftarrow v\}$$

  for each edge $\{u, v\} \in E$.
- For all $v \in V$ define

$$\begin{aligned} \text{in-deg}(v, \sigma) &= |\{u : (u, v) \in \sigma(E)\}| \\ \text{out-deg}(v, \sigma) &= |\{u : (v, u) \in \sigma(E)\}| \\ \deg(v) &= \text{in-deg}(v, \sigma) + \text{out-deg}(v, \sigma). \end{aligned}$$

- A node v with $\text{in-deg}(v, \sigma) = \deg(v)$ is called a *sink*. We call an orientation σ *sinkless* if no node is a sink, that is, every node v has $\text{out-deg}(v, \sigma) > 0$.

Sinkless orientation

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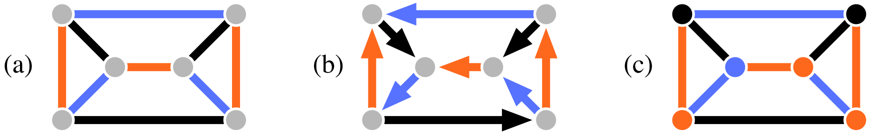

Figure: (a) A 3-regular edge 3-coloured graph. (b) A sinkless orientation. (c) A sinkless colouring.

Colourings

- We write $[k] = \{0, 1, \dots, k-1\}$.
- $\psi\colon E \to [\chi]$ is a *proper edge χ-colouring* if any two adjacent edges have a different colour.
- Given a properly edge χ-coloured graph $G = (V, E, \psi)$, we call $\varphi\colon V \to [\chi]$ a *sinkless colouring* of G if for all edges $e = \{u, v\} \in E$ it holds that
$$\varphi(u) = \psi(e) \Rightarrow \varphi(v) \neq \psi(e),$$
that is, if at least one endpoint of each edge has a different colour than the edge.

Sinkless colouring

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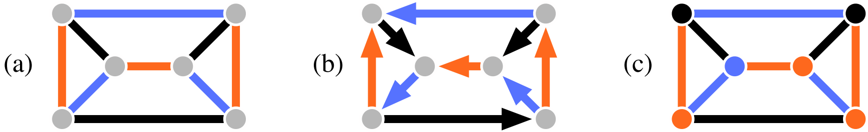

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Graph problem definitions

Problem (Sinkless colouring)

Given an edge d-coloured d-regular graph $G = (V, E, \psi)$, find a sinkless colouring φ. That is, compute a colouring φ such that for no edge $e = \{u, v\} \in E$ we have $\varphi(u) = \varphi(v) = \psi(e)$.

Problem (Sinkless orientation)

Given an edge d-coloured d-regular graph $G = (V, E, \psi)$, find a sinkless orientation. That is, compute an orientation σ such that $\text{out-deg}(v, \sigma) > 0$ *for all $v \in V$.*

Relationship between the graph problems

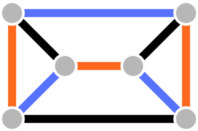
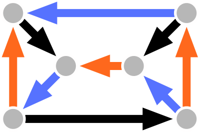
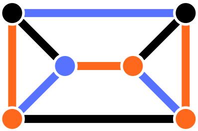
(a)  (b)  (c) 

Figure: (a) A 3-regular edge 3-coloured graph. (b) A sinkless orientation. (c) A sinkless colouring.

- From a sinkless orientation we get a sinkless colouring in 0 rounds.
- From a sinkless colouring we get a sinkless orientation in 1 rounds.

From LLL to sinkless orientation

Theorem

Let $f : \mathbb{N} \to \mathbb{R}$ be such that $f(4) \leq 16$. Let A be a Monte Carlo distributed algorithm for LLL such that A finds an assignment avoiding all the bad events under the LLL criteria $pf(d) \leq 1$ in time T for some $T : \mathbb{N} \to \mathbb{N}$. Then there is a Monte Carlo distributed algorithm B that finds a sinkless orientation in 3-regular graphs of girth at least 5 in time $O(T)$.

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- We start with 4-regular graphs $G = (V, E)$.
- Set $\mathrm{vbl}(E_v) = \{X_e \ : \ v \in e\}$ for each $v \in V$
- For each $e = \{u, v\} \in E$, the variable X_e ranges over $\{u \to v, u \leftarrow v\}$
- The bad event E_v occurs exactly when for all neighbours u of v the variable $X_{\{v,u\}}$ takes the value $u \to v$

From LLL to sinkless orientation

- If the variables X_e are sampled uniformly at random, we have $\Pr(E_v) = 1/2^4 = 1/16$ for each $v \in V$.
- Let $p = 1/16$ and $d = 4$. Now $\Pr(E_v) \leq p$ and E_v depends on d other events for each $v \in V$, and the condition $pf(d) \leq 1$ holds, given $f(4) \leq 16$.
- Run the algorithm A and define an orientation σ of G by setting $\sigma(e) = a_v(X_e)$, where $v \in e$, for each $e \in E$.
- Now σ is a sinkless orientation of G.

From 3-regular to 4-regular graphs

- LLL is not directly applicable: the probability of bad events would be $p = 1/2^3 = 1/8$ and thus $ep(d+1) > 1$.
- *Contract* edges of one colour class to obtain a 4-regular graph.
- Simulate the algorithm for the 4-regular case in the 3-regular graph.

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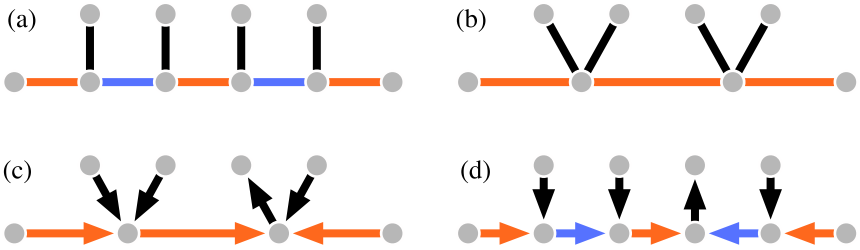

The mutual speedup lemma

Lemma

Suppose B is a sinkless colouring algorithm that runs in t rounds such that for any edge $e = \{u, v\}$ the probability of outputting a forbidden configuration $B(u) = \psi(e) = B(v)$ is at most p. Then there exists a sinkless orientation algorithm B' that runs in t rounds such that for any node u the probability of being a sink is at most $6p^{1/3}$.

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Lemma

Suppose B' is a sinkless orientation algorithm that runs in time t such that the probability that any node u is a sink is at most ℓ. Then there exists a sinkless colouring algorithm B'' that runs in time $t - 1$ such that the probability for any edge $e = \{u, v\}$ having a forbidden configuration $B''(u) = \psi(e) = B''(v)$ is less than $4\ell^{1/4}$.

From colouring to orientation: the proof idea

- Given a randomised sinkless *colouring* algorithm B running in t rounds, construct a randomised sinkless *orientation* algorithm B' that also runs in t rounds.
- We write $B(u)$ for the colour that u outputs according to B and $B'(e)$ for the orientation B' outputs for edge e.
- We denote the radius-t neighbourhood of a node u by

$$N^t(u) = \{v \in V \, : \, \mathrm{dist}(u, v) \leq t\},$$

  where $\mathrm{dist}(u, v)$ is the length of the shortest path between u and v.
- The radius-t neighbourhood of an *edge* $\{u, v\}$ is

$$N^t(\{u, v\}) = N^t(u) \cap N^t(v).$$

From colouring to orientation: the proof idea

Consider any node $u \in V$. Algorithm B' consists of three steps:

1. Node u gathers its radius-t neighbourhood $N^t(u)$ in t rounds.
2. Node u computes the set $C(u)$ of *candidate colours*:

$$C(u) = \left\{ \psi(e) \ : \ \Pr[B(u) = \psi(e) \mid N^t(e)] \geq p^{1/3} \text{ and } e = \{u, v\} \right\},$$

   In addition, for each $e = \{u, v\}$ node u calculates the probability of v outputting the colour $\psi(e)$ when executing B. Thus u can determine whether $\psi(e) \in C(v)$.

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3. If $\psi(e) \in C(u) \cap C(v)$ or $\psi(e) \notin C(u) \cup C(v)$, choose the orientation $B'(e)$ of edge e arbitrarily. Otherwise, edge e is oriented according to the following rule:

   $$B'(e) = \begin{cases} u \to v & \text{if } \psi(e) \in C(u) \text{ and } \psi(e) \notin C(v), \\ u \leftarrow v & \text{if } \psi(e) \notin C(u) \text{ and } \psi(e) \in C(v). \end{cases}$$

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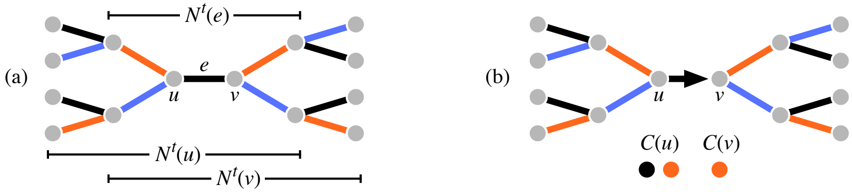

- Here the running time $t = 2$.
- In algorithm B, the colour of node u is determined by the random bits in $N^t(u)$.
- Black is a candidate colour of u if, based on the information in $N^t(e)$, the probability of u outputting black in algorithm B is at least $p^{1/3}$.
- If black is one of the candidate colours of u, and it is *not* one of the candidate colours of v, then algorithm B' will orient the edge $u \to v$.

Conclusion

- There is a randomised distributed algorithm for LLL that runs in $O(\log n)$ rounds in bounded-degree graphs.
- The best previously known lower bound was $\Omega(\log^* n)$ rounds.
- We show that any randomised Monte Carlo algorithm for LLL that finds a satisfying assignment with high probability requires $\Omega(\log \log n)$ rounds.